# Signed Graphs and Applications to Social Science

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# Presentation Flow

- Signed Graphs and History
- The Structure Theorem
- The Cycle Basis
- Addressing Weaknesses in the Model
- Application 1: Coalitions in Congress
- Application 2: Games on Signed Graphs

# What is a signed graph?

- A signed graph $G(V, E, \sigma)$ is a graph $G = (V, E)$ together with a function $\sigma : E \to \{+, -\}$ which attaches a sign to each edge [2]
  - Motivation: additional structure
- Signed graphs represent friendly or antagonistic relationships
- Applications to social science
  - Mapping partisan polarization in Congress
  - Resource allocation game in international relations

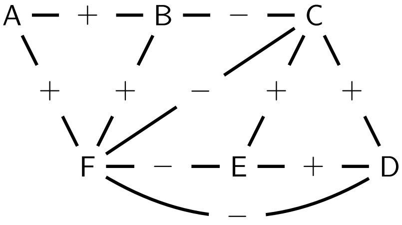
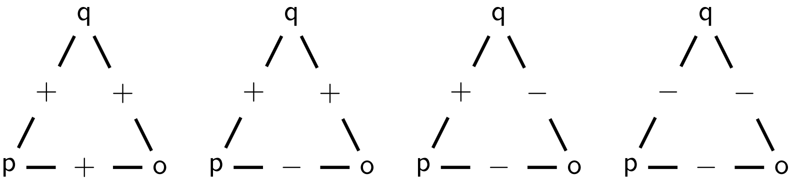



Figure: Example Signed Graph

# History and Structural Balance Theory

- From Sociology to Mathematics
  - Logical construction due to Heider (1946) [3]
  - Generalization to graphs by Harary and Cartwright (1956) [6]
- Structural Balance Theory
  - Describe how local relationships impact global network behavior
    - Cognitive dissonance creates stress; move toward a balanced equilibrium [3]
    - *"The enemy of my enemy is my friend"*
  - Sign of a subgraph = product of the edges




(a) Balanced (b) Unbalanced (c) Balanced (d) Unbalanced

Figure: Heider's Triads

- **Theorem 1**
  **(Structure Theorem)** [6]:
  Suppose $G$ is a signed graph. Then the following are equivalent:
  1. $G$ is balanced.
  2. Every closed chain in $G$ is positive.
  3. Any two chains between vertices $u$ and $v$ have the same sign.
  4. The set $V$ can be partitioned into two sets $A$ and $B$ such that every positive edge joins vertices of the same set and every negative edge joins vertices of different sets.

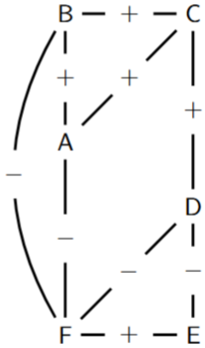



Figure: Signed graph $G$

# Balance as a Bipartition (Statement 4)

- Positive edges are omitted to preserve bipartite structure

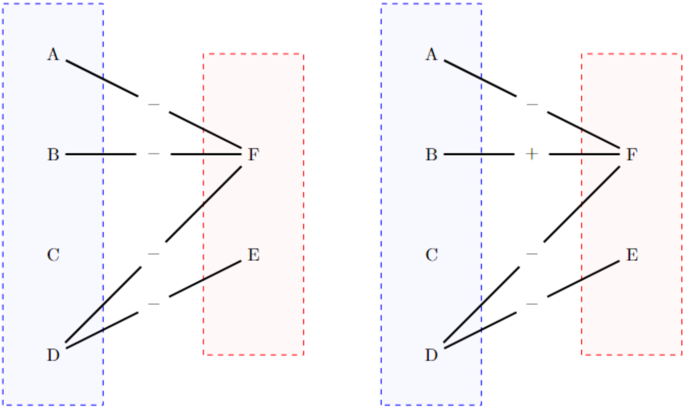


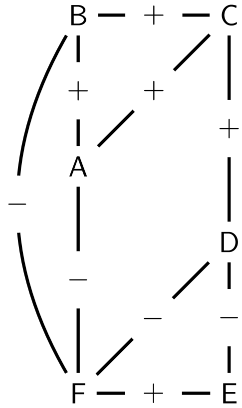

Figure: Example of a balanced and unbalanced bipartition

# Enter: The Cycle Basis

- Statement 2 of the Structure Theorem posits that a balanced graph must have all positive closed chains (cycles)
  - **Definition:** The cycle space $C(G)$ of a connected graph $G$ is the collection of its Eulerian spanning subgraphs.[5]
  - Tedious to check all cycles; need a better tool to verify
- Intuition: what if we check only basis elements instead of every cycle?
  - **Definition:** A spanning tree in a graph G is a subgraph of G that includes every vertex of G and is also a tree.[5]
  - **Theorem 2 (Cycle Basis)** [5]:
    Let $G$ be a graph, and let $C(G)$ be its cycle space. Let $T$ be a spanning tree of $G$, and for each edge $e$ not in $T$, let $C_e$ be the unique cycle in $G$ that contains $e$ and has all other edges in $T$. Then $B = \{C_e \mid e \notin T\}$ is a basis for $C(G)$.

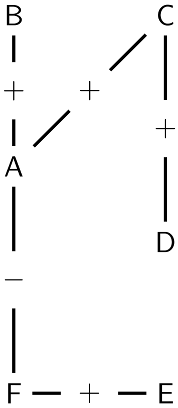
# Cycle Basis Example

- Note: $T$ is not unique




(a) Signed Graph $G$

$\longrightarrow$




(b) Spanning Tree $T$

# Cycle Basis Example Continued

- To construct the cycle basis, find each unique cycle created by attaching edges $e = E_G \backslash E_T$ to $T$

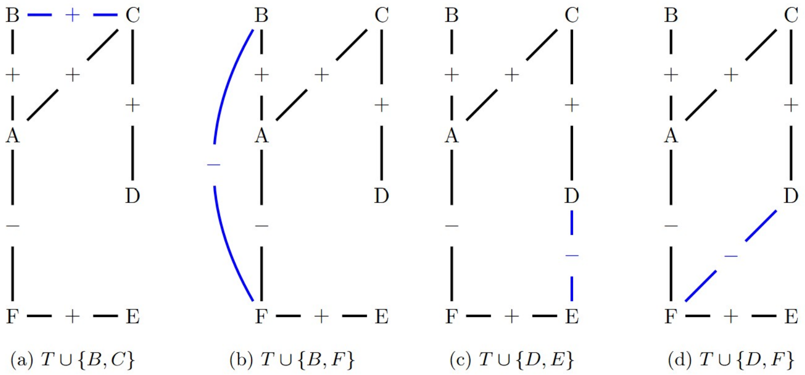



Figure: Attaching missing edges $e$ to $T$

# Cycle Basis Example Continued

- The cycle basis is:
  - Minimal
  - Not unique
- To verify that a graph is balanced, it's enough to simply verify balance of each basis element

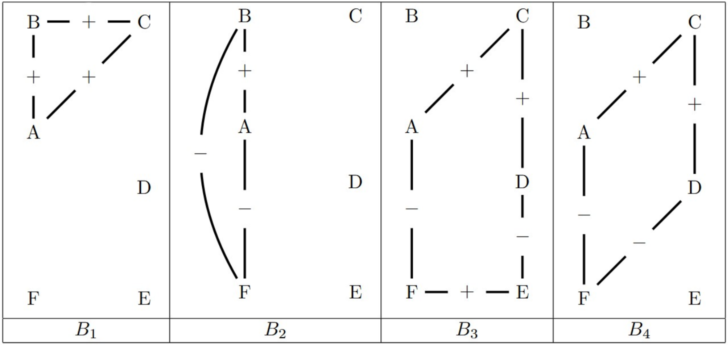



Figure: Cycle Basis $B$ of $C(G)$

# Cycle Basis Example Continued

- With the basis elements defined, one can create any cycle in the graph with a combination of basis elements
  - **Definition:** Given $C_i,\ C_j \in C(G)$, the symmetric difference is defined to be $C_i \oplus C_j = (C_i \setminus C_j) \cup (C_j \setminus C_i)$.

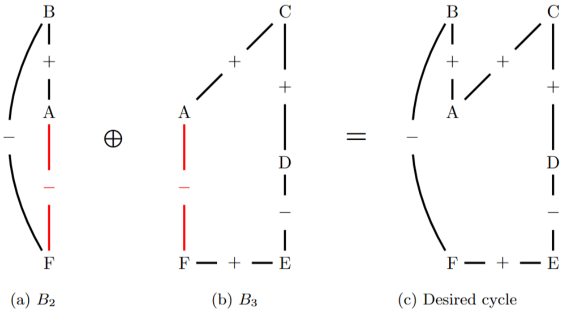
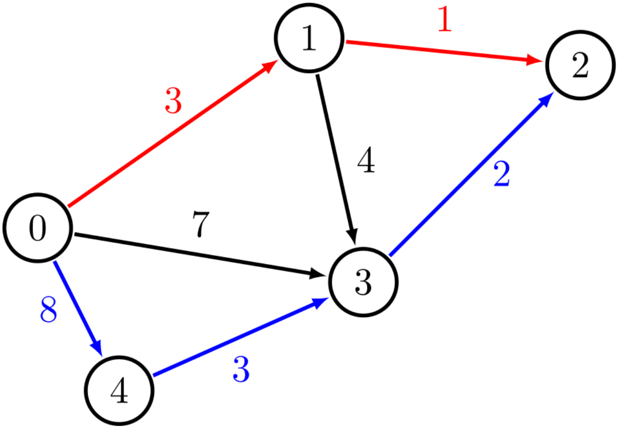



Figure: Computing a cycle from two basis elements

# Addressing Weaknesses

- Weighted
- Directed
- Weak balance for n-partite graphs
- Partial balance



# Application 1: Partisan Polarization in Congress

- Paper: *Detecting coalitions by optimally partitioning signed networks of political collaboration*, Samin Aref and Zachary Neal (2020) [1]
- Balance can serve as a measure for partisan polarization: a balanced graph is completely polarized
- The authors count the number of co-authored bills between any two legislators and compare against the null
  - If the count is lower than the null, connect with a negative edge
  - If the count is greater than the null, connect with a positive edge
- However, the existence bipartisan bills violate the strictly negative bipartition between groups which defines balance

# Partisan Polarization in Congress, Continued

- Instead, measure partial balance with the Triangle Index and Frustration Index
  - The Triangle Index $T(G)$ is the fraction of positive 3-cycles in the graph
  - The Frustration Index $F(G)$ measures the number of positive edges between groups

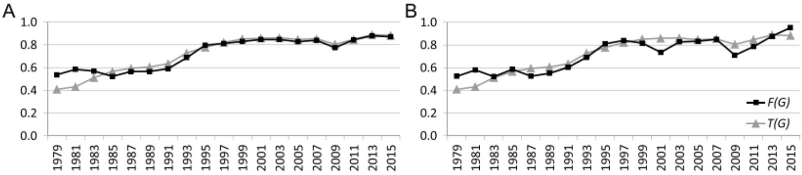



**Figure 2.** Two measures of partial balance indicating an overall increase in political polarization in (**A**) US House of Representatives and (**B**) US Senate over the time period 1979–2016.

# Application 2: The Power Allocation Game

- Paper: *Games on Signed Graphs*, Yuke Li and A. Stephen Morse (2022) [4]
- In International Relations, countries interact with each other in an adversarial networked environment: alliance ties determine where resources get allocated
- Countries may choose to allocate some of their wealth to their friends or to the destruction of their enemies
- Primitives:
  - Country index $\mathbf{n} = \{1, 2, ..., n\}$
  - An environment $\mathbb{E}_E = \{\mathcal{V}, \mathcal{E}_E\}$
  - $\mathcal{F}_i = \{\text{friends of country i}\}$
  - $\mathcal{A}_i = \{\text{adversaries of country i}\}$
  - $p_i =$ power of country i, measured in currency
  - An alliance $\mathcal{S}$ with members $j \in \mathcal{S} \subset \mathbf{n}$ with $j \in \mathcal{F}_i$
  - Enemies of the alliance $\mathcal{A}_\mathcal{S} = \bigcup_{i \in \mathcal{S}} \mathcal{A}_i$

# The Power Allocation Game, Continued

- Countries allocate $p_i$ via a $(1 \times n)$ strategy vector $u_{ij}$ subject to $u_{i1} + \ldots + u_{in} = p_i$.
- Allocations are represented with a directed graph $\mathbb{E}_A = \{\mathcal{V}, \mathcal{E}_A\}$ which has the weights $u_{ij}$; negative signs are attached to enemies and positive signs are attached to friends
- A country will survive if $p_i \geq \sum_{j \in A_i} p_j$ (their power is at least equal to all adversaries) or if $\sum_{i \in \mathcal{S}} p_i \geq \sum_{j \in \mathcal{S}_\mathcal{A}} p_j$ (they are a part of an alliance which matches the power of adversary alliances)
- Takeaways:
  - Countries pursue "survival and success in a constant or a changing environment, and may bring about some of the changes to the environment itself." [4]
  - Countries are constrained both by their resources and their relationships

# References


[1] Samin Aref and Zachary Neal, *Detecting coalitions by optimally partitioning signed networks of political collaboration*, Scientific reports **10** (2020), no. 1, 1506.

[2] Abdelhakim El Maftouhi, Ararat Harutyunyan, and Yannis Manoussakis, *Weak balance in random signed graphs*, Internet Mathematics **11** (2015), no. 2, 143–154.

[3] Fritz Heider, *Attitudes and cognitive organization*, The Journal of psychology **21** (1946), no. 1, 107–112.

[4] Yuke Li and A Stephen Morse, *Games on signed graphs*, Automatica **140** (2022), 110243.

[5] Mary Radcliffe, *Cycle bases*, Carnegie Mellon, Department of Mathematical Sciences, 2018.

[6] Fred S. Roberts, *Discrete mathematical models, with applications to social, biological, and environmental problems*, (No Title) (1976).

# End Frame

The presentation is over.